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Research Article

The Existence of Solutions for Impulsive Fractional Partial Neutral Differential Equations

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This paper deals with the existence of mild solutions of a class of impulsive fractional partial neutral semilinear differential equations. A series of analytical results about the mild solutions are obtained by using fixed-point methods. Then, we present an example to further illustrate the applications of these results.

1. Introduction

Great progress in fractional differential equations has been achieved in recent years. Due to their broad applications in science and engineering such as physics, economics, biology, and mechanical engineering, fractional differential equations attract attention of researchers from different areas. Compared with ordinary differential equation systems and partial differential equation systems, fractional differential equation systems have the potential to model real-world problems with more accuracy. In order to further investigate these models, it is essential to study the fractional differential equations analytically. Though, mathematically, a fractional differential equation is closely related to its corresponding ordinary differential equation or partial differential equation, that is, the ordinary differential equation or the partial differential equation can be obtained by letting $\alpha = 1, 2, \ldots$ in its corresponding fractional differential equation, many mathematical methods which can be used in investigating ordinary differential equations or partial differential equations fail in analyzing fractional differential equations since fractional differential equations usually have more complicated structures and different properties. Thus, it is essential to develop novel methods to study fractional differential equations analytically. With the aim of analyzing fractional differential equations, extensive investigations had been carried out [1–15].

In applied mathematics, another two types of differential equations, impulsive differential equations and functional differential equations with different conditions have wide applications in modeling particular phenomena and dynamical processes in physics, automatics, robotics, biology, medicine, and so forth. For example, impulsive differential equations can be used to investigate natural phenomena or dynamical processes which are subject to great changes in a short period of time [16–18]. To further understand the mathematical models with impulsive differential equations or functional differential equations with different conditions, analytical investigation of these equations is required. During the last ten years, progress in studying these two types of differential equations has been made [19–25].

One of the important techniques for analyzing impulsive differential equations is the semigroup theory, which has been successfully used in investigating the existence, uniqueness, and continuous dependence of the solutions of impulsive differential equations. It also has wide applications in the study of periodic and almost periodic solutions of different kinds of differential equations. For example, Fan and Li studied the existence results for semilinear differential equations with nonlocal and impulsive conditions using semigroup theory [26]. The theory has also been used to investigate a mixed monotone iterative technique for a class of semilinear impulsive evolution equations in Banach spaces [27]. With the aid of the semigroup theory, a series of conclusions about the existence of the solutions of functional differential equations and functional integral equations have been obtained [19–23, 28–36].

Since the fractional derivative does not satisfy the Leibniz rule and the corresponding fractional operator equation does not satisfy the variation of constants formula [37], which implies that the corresponding operator does not satisfy the properties of semigroups (see [38]), $T_\alpha(t) \triangleq t^{\alpha-1}E_{\alpha,\alpha}(At^\alpha)$ does not satisfy the semigroup relations, that is, $T_\alpha(t+s) \neq T_\alpha(t)T_\alpha(s)$, indicating the properties of corresponding resolvent operator. It is difficult to define the mild solutions of fractional partial differential evolution equations. In particular, semigroup theory was used inappropriately to study the existence and uniqueness of mild solutions to impulsive fractional differential equations [13] and impulsive partial neutral functional differential equation [39]. The existence of mild solutions for a class of impulsive fractional partial semilinear differential equations was investigated in [15] with errors in [13] corrected.

A solution to this problem has been given in [15], in which the mild solutions of impulsive fractional differential equations are defined using piecewise functions, and a sufficient condition, which guarantees the existence and uniqueness of solutions to the equations, is obtained. The existence of solutions to a fractional neutral integrodifferential equation with unbounded delay was studied in [40].

In this paper, with the aim of investigating the existence theorem for the solutions of impulsive fractional neutral functional differential equations, we first define the mild solution of fractional neutral functional differential equation using Laplace transform. Then, by introducing the operator $S_{\alpha,k,l}(t)$, the mild solution of impulsive fractional differential equation is defined. Through analyzing operator $S_\alpha(t)$, $T_\alpha(t)$, and the corresponding semigroup $T(t)$, a sufficient condition, which guarantees the existence and uniqueness of solutions to the following system, is obtained:

$$\begin{aligned} &D_t^\alpha (x(t) + F(t,x_t)) + Ax(t) = G(t,x_t), \\ &\qquad t \in I = [0,T],\ t \neq t_k, \\ &x(0) = \varphi \in \mathscr{B}, \Delta x|_{t=t_k} = I_k\left(x_{t_k^-}\right), \\ &\qquad k = 1,\ldots,m, \end{aligned} \tag{1}$$

where D_t^α, for $0 < \alpha \leq 1$, is the Caputo fractional derivative, $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$, and $(X, \|\cdot\|)$ is a Banach space.

The rest of this paper is organized as follows. In Section 2, we present some notations, definitions, and theorems which will be used in the following sections. In Section 3, the definition of mild solution of the system (1) and the relation between analytic semigroup $T(t)$ and some solution operators are given. The main results of our paper are given in Section 4. In Section 5, application of the obtained results is presented.

2. Preliminaries

In this section, we will introduce some notations, definitions, and theorems, which will be used throughout this paper. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be Banach spaces, and let $\mathscr{L}(Y,X)$ be the Banach space of bounded linear operators from Y into X equipped with its natural topology. When $Y = X$, we use the notation $\mathscr{L}(X)$ to denote the Banach space. We say that a function $u(\cdot)$ is a normalized piecewise continuous function on $[\mu,\tau]$ if u is piecewise continuous, and left-continuous on $(\mu,\tau]$. The space formed by the normalized piecewise continuous functions from $[\mu,\tau]$ to X is denoted by $\mathscr{PC}([\mu,\tau];X)$, where the notation $\mathscr{PC}$ stands for the space formed by all functions $u \in \mathscr{PC}([0,\mu];X)$ such that $u(\cdot)$ is continuous at $t \neq t_i$, and $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exist for all $i = 1,\ldots,m$. In addition, we use $(\mathscr{PC}, \|\cdot\|_{\mathscr{PC}})$ to denote the space $\mathscr{PC}$ endowed with the norm $\|x\|_{\mathscr{PC}} = \sup_{s\in I}\|x(s)\|$. Obviously, $(\mathscr{PC}, \|\cdot\|_{\mathscr{PC}})$ is a Banach space. Moreover, an axiomatic definition for the phase space $\mathscr{B}$ is employed. Here, the definition of $\mathscr{B}$ is similar to that given in [41]. In particular, $\mathscr{B}$ is a linear space of functions mapping $(-\infty,0]$ into X endowed with a seminorm $\|\cdot\|_{\mathscr{B}}$ and the following axioms hold.

(I) If $x : (-\infty,\mu+b] \to X$, $b > 0$, such that $x_\mu \in \mathscr{B}$ and $x|_{[\mu,\mu+b]} \in \mathscr{PC}([\mu,\mu+b] : X)$, then for every $t \in [\mu,\mu+b]$ the following conditions hold:

- (i) x_t is in $\mathscr{B}$;
- (ii) $\|x(t)\| \leq H\|x_t\|_{\mathscr{B}}$;
- (iii) $\|x_t\|_{\mathscr{B}} \leq K(t-\mu)\sup\{\|x(s)\| : \mu \leq s \leq t\} + M(t-\mu)\|x_\mu\|_{\mathscr{B}}$,

where $H > 0$ is a constant, $K, M : [0,\infty) \to [1,\infty)$, K is continuous, M is locally bounded, and H, K, M are independent of $x(\cdot)$.

(II) The space $\mathscr{B}$ is complete.

Next, we give an example to illustrate the above definitions.

Example 1 (the phase space $\mathscr{PC}_r \times L^2(g,X)$). Let $r > 0$ and $g : (-\infty,-r) \to R$ be a nonnegative, locally Lebesgue integrable function. Assume that there is a non-negative measurable, locally bounded function $\eta(\cdot)$ on $(-\infty,0]$ such that $g(\xi+\theta) \leq \eta(\xi)g(\theta)$ for all $\xi \in (-\infty,0]$ and $\theta \in (-\infty,-r] \setminus N_\xi$, where $N_\xi \subset (-\infty,-r)$ is a set with Lebesgue measure zero. $\mathscr{PC}_r \times L^2(g,X)$ denotes the set of all functions $\varphi : (-\infty,0] \to X$ such that $\varphi|_{[-r,0]} \in \mathscr{PC}([-r,0],X)$ and $\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|_X^2 d\theta < \infty$. In $\mathscr{PC}_r \times L^2(g,X)$, we consider the seminorm defined by

$$\|\varphi\|_{\mathscr{B}} = \sup_{\theta\in[-r,0]} \|\varphi(\theta)\|_X + \left(\int_{-\infty}^{-r} g(\theta)\|\varphi(\theta)\|_X^2 d\theta\right)^{1/2}. \tag{2}$$

The preceding conditions indicate that the space $\mathscr{PC}_r \times L^2(g,X)$ satisfies the axioms (I) and (II). Moreover, when $r = 0$, we can take $H = 1$, $K(t) = \left(\int_{-t}^0 g(\theta)\right)^{1/2}$, and $M(t) = \eta(-t)$ for $t \geq 0$.

Definition 2. Let $a, \alpha \in \mathbb{R}$. A function $f : [a, \infty) \to X$ is said to be in the space $C_{a,\alpha}$ if there exist a real number $p > \alpha$ and a function $g \in C([a, \infty), X)$ such that $f(t) = t^p g(t)$. Moreover, f is said to be in the space $C^m_{a,\alpha}$ for some positive integer m if $f^{(m)} \in C_{a,\alpha}$.

Definition 3. If the function $f \in C^m_{a,-1}$, where $m \in \mathbb{N}^+$, the fractional derivative of order $\alpha > 0$ of f in the Caputo sense is defined as

$$D^\alpha_t f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} f^{(m)}(s)\, ds, \quad m-1 < \alpha \le m. \tag{3}$$

Definition 4. Let $A : \mathscr{D} \subseteq X \to X$ be a closed linear operator. A is said to be sectorial if there exist $0 < \theta < \pi/2$, $M > 0$, and $\mu \in \mathbb{R}$ such that the resolvent of A exists outside the sector

$$\begin{gathered} \mu + S_\theta = \{\mu + \lambda : \lambda \in \mathbb{C}, |\arg(-\lambda)| < \theta\}, \\ \left\|(\lambda I - A)^{-1}\right\| \le \frac{M}{|\lambda - \mu|}, \quad \lambda \notin \mu + S_\theta \end{gathered} \tag{4}$$

(for short, we say that A is sectorial of type (M, θ, μ)).

Let $-A$ denote the infinitesimal generator of an analytic semigroup in a Banach space and $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of $\rho(A)$. We define the fractional power A^{-q} by

$$A^{-q} = \frac{1}{\Gamma(q)} \int_0^\infty (t)^{q-1} T(t)\, dt, \quad q > 0. \tag{5}$$

For $0 < q \le 1$, $A^q = (A^{-q})^{-1}$ is a closed linear operator whose domain $D(A^q) \supset D(A)$ is dense in X. For analytic semigroup $\{T(t)\}_{t\ge0}$, we have the following theorem.

Theorem 5 ([42, Theorem 6.13]). *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge0}$. If $0 \in \rho(A)$, then the following statements hold.*

(a) *$T(t) : X \to D(A^\alpha)$ for every $t > 0$ and $\alpha \ge 0$.*

(b) *For every $x \in D(A^\alpha)$, we have $T(t)A^\alpha x = A^\alpha T(t) x$.*

(c) *For every $t > 0$, the operator $A^\alpha T(t)$ is bounded and*

$$\|A^\alpha T(t)\| \le M_\alpha t^{-\alpha} e^{-\delta t}. \tag{6}$$

(d) *$0 < \alpha \le 1$ and $x \in D(A^\alpha)$, then*

$$\|T(t)x - x\| \le c_\alpha t^\alpha \|A^\alpha x\|. \tag{7}$$

3. Mild Solutions

In this section, we investigate the classical solution of the following equation:

$$\begin{gathered} D^\alpha_t (x(t) + F(t, x_t)) + Ax(t) = G(t, x_t), \\ t \in I, \ 0 < \alpha \le 1, \\ x(0) = \varphi \in \mathscr{B}. \end{gathered} \tag{8}$$

Based on the classical solution, the mild solution of system (1) is defined. Then, the relationship between the analytic semigroup $T(t)$ and some solution operators is given.

Lemma 6. *Let $-A$ be the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge0}$. If G and F satisfy the uniform Hölder condition with exponent $\beta \in (0, 1]$, then the solutions of the Cauchy problem (8) are fixed points of the operator equation*

$$\Psi x(t) = \begin{cases} \mathscr{S}_\alpha(t)(\varphi(0) + F(0, \varphi)) - F(t, x_t) \\ \quad - \displaystyle\int_0^t T_\alpha(t-s) AF(s, x_s)\, ds \\ \quad + \displaystyle\int_0^t T_\alpha(t-s) G(s, x_s)\, ds, \\ x_0 = \varphi \in \mathscr{B}, \end{cases} \tag{9}$$

where

$$\begin{gathered} \mathscr{S}_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} \lambda^{\alpha-1} R(\lambda^\alpha, -A)\, d\lambda, \\ T_\alpha(t) = \frac{1}{2\pi i} \int_c e^{\lambda t} R(\lambda^\alpha, -A)\, d\lambda, \end{gathered} \tag{10}$$

with c being a suitable path such that $\lambda^\alpha \notin \mu + S_\theta$ for $\lambda \in c$.

Proof. Applying the Laplace transform to (9), we obtain

$$\begin{aligned} &\lambda^\alpha (\mathscr{L}x(t))(\lambda) + \lambda^\alpha (\mathscr{L}F(t, x_t))(\lambda) \\ &\quad - \lambda^{\alpha-1}(\varphi(0) + F(0, \varphi)) \\ &\quad + A(\mathscr{L}x)(\lambda) = (\mathscr{L}G(t, x_t))(\lambda). \end{aligned} \tag{11}$$

It follows that

$$\begin{aligned} (\mathscr{L}x)(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha I + A)^{-1}(\varphi(0) + F(0, \varphi)) \\ &\quad - \lambda^\alpha(\lambda^\alpha + A)^{-1}(\mathscr{L}F(t, x_t))(\lambda) \\ &\quad + (\lambda^\alpha I + A)^{-1}(\mathscr{L}G(t, x_t))(\lambda). \end{aligned} \tag{12}$$

Noting that $\lambda^\alpha(\lambda^\alpha + A)^{-1} = I - A(\lambda^\alpha + A)^{-1}$, and applying the inverse Laplace transform, we have

$$\begin{aligned} x(t) &= \mathscr{S}_\alpha(t)(\varphi(0) + F(0, \varphi)) - F(t, x_t) \\ &\quad + \int_0^t T_\alpha(t-s) AF(s, x_s)\, ds \\ &\quad + \int_0^t T_\alpha(t-s) G(s, x_s)\, ds. \end{aligned} \tag{13}$$

Since F and G satisfy a uniform Hölder condition, with exponent $\beta \in (0, 1)$, then the classical solutions of Cauchy problem (8) are fixed points of the following operator equations (see [43]):

$$\begin{aligned} \Psi x(t) &= \mathscr{S}_\alpha(t)(\varphi(0) + F(0, \varphi)) - F(t, x_t) \\ &\quad + \int_0^t T_\alpha(t-s) AF(s, x_s)\, ds \\ &\quad + \int_0^t T_\alpha(t-s) G(s, x_s)\, ds. \end{aligned} \tag{14}$$

□

In what follows, we use the notations $\mathcal{S}_{\alpha,k,l}(t)x \triangleq \mathcal{S}_\alpha(t-t_k)\circ\mathcal{S}_\alpha(t_k-t_{k-1})\circ\cdots\circ\mathcal{S}_\alpha(t_{l+1}-t_l)x$, $0\le l\le k-1$, $t_0=0$, for all $x\in X$, and $\overline{\mathcal{S}_{\alpha,k,l}}(t)x \triangleq \mathcal{S}_\alpha(t-t_k)\circ\mathcal{S}_\alpha(t_k-t_{k-1})\circ\cdots\circ\mathcal{S}_\alpha(t_{l+2}-t_{l+1})x$, $0\le l\le k-1$.

Definition 7. A function $u\in\mathscr{PC}(I,X)$ is called a mild solution of (1) if it satisfies the operator equation

$$\begin{aligned} u(t) &= \mathcal{S}_{\alpha,k,0}(t)(\varphi(0)+F(0,\varphi)) - F(t,u_t) \\ &\quad + \sum_{0<t_{l+1}<t,0\le l\le k}\overline{\mathcal{S}_{\alpha,k,l}}(t) \\ &\quad \times \int_{t_l}^{t_{l+1}} AT_\alpha(t_{l+1}-s)F(s,u_s)\,ds \\ &\quad + \int_{t_k}^{t} AT_\alpha(t-s)F(s,u_s)\,ds \\ &\quad + \sum_{0<t_{l+1}<t,0\le l\le k}\overline{\mathcal{S}_{\alpha,k,l}}(t) \\ &\quad \times \int_{t_l}^{t_{l+1}} T_\alpha(t_{l+1}-s)G(s,u_s)\,ds \\ &\quad + \int_{t_k}^{t} T_\alpha(t-s)G(s,u_s)\,ds \\ &\quad + \sum_{0<t_{l+1}<t,0\le l\le k}\overline{\mathcal{S}_{\alpha,k,l}}(t) \\ &\quad \times\left[I_{l+1}(u_{t_{l+1}}) + \Delta F(t_{l+1},u_{t_{l+1}})\right], \quad t\in I. \end{aligned} \tag{15}$$

Remark 8. It is easy to verify that a classical solution of (1) is a mild solution of the same system. Thus, Definition 7 is well defined (see [42, 43]).

Lemma 9. *If $-A$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\ge0}$ and $0\in\rho(A)$, then we have*

$$\begin{aligned} T_\alpha(t) &= \alpha\int_0^\infty \theta\phi_\alpha(\theta)t^{\alpha-1}T(t^\alpha\theta)\,d\theta, \\ \mathcal{S}_\alpha(t) &= \int_0^\infty \phi_\alpha(\theta)T(t^\alpha\theta)\,d\theta, \end{aligned} \tag{16}$$

where $\phi_\alpha(\theta)$ is the probability density function defined on $(0,\infty)$ such that its Laplace transform is given by

$$\int_0^\infty e^{-\theta x}\phi_\alpha(\theta)\,d\theta = \sum_{j=0}^{\infty}\frac{(-x)^j}{\Gamma(1+\alpha j)}, \quad x>0, \tag{17}$$

and satisfies

$$\begin{aligned} \int_0^\infty \phi_\alpha(\theta)\,d\theta &= 1, \\ \int_0^\infty \theta^\eta\phi_\alpha(\theta)\,d\theta &\le 1, \quad 0\le\eta\le1. \end{aligned} \tag{18}$$

Proof. For all $x\in D(A)\subset X$, we have

$$(\lambda+A)^{-1}x = \int_0^\infty e^{-\lambda s}T(s)\,x\,ds. \tag{19}$$

Let

$$\int_0^\infty e^{-\lambda\theta}\psi_\alpha(\theta)\,d\theta = e^{-\lambda^\alpha}, \tag{20}$$

where $\alpha\in(0,1)$, $\psi_\alpha(\theta) = (1/\pi)\Sigma_{1\le n<\infty}(-1)^n\theta^{-\alpha n-1}(\Gamma(n\alpha+1)/n!)\sin(n\pi\alpha)$, and $\theta\in(0,\infty)$ (see [44]). Thus, we obtain

$$\begin{aligned} &(\lambda^\alpha+A)^{-1}x \\ &= \int_0^\infty e^{-\lambda^\alpha s}T(s)\,x\,ds \\ &= \int_0^\infty\int_0^\infty \alpha t^{\alpha-1}e^{-(\lambda t)^\alpha}T(t^\alpha)\,x\,dt \\ &= \int_0^\infty\int_0^\infty \alpha t^{\alpha-1}e^{-\lambda t\theta}\psi_\alpha(\theta)T(t^\alpha)\,x\,d\theta\,dt \\ &= \int_0^\infty\int_0^\infty \alpha\psi_\alpha(\theta)e^{-\lambda t}T\left(\frac{t^\alpha}{\theta^\alpha}\right)\frac{t^{\alpha-1}}{\theta^\alpha}\,dt\,d\theta \\ &= \int_0^\infty e^{-\lambda t}\left(\alpha\int_0^\infty\theta\phi_\alpha(\theta)t^{\alpha-1}T(t^\alpha\theta)\,x\,d\theta\right)dt, \end{aligned} \tag{21}$$

where $\phi_\alpha(\theta) = (1/\alpha)\theta^{-1-1/\alpha}\psi_\alpha(\theta^{-1/\alpha})$ is the probability density function defined on $(0,\infty)$ and satisfies $\int_0^\infty\phi_\alpha(\theta)d\theta = 1$ and $\int_0^\infty\theta^\eta\phi_\alpha(\theta)d\theta\le1$, $0\le\eta\le1$ (see [44]). Then, from (21), we get

$$\begin{aligned} T_\alpha(t) &= \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda^\alpha+A)^{-1}d\lambda \\ &= \alpha\int_0^\infty\theta\phi_\alpha(\theta)t^{\alpha-1}T(t^\alpha\theta)\,d\theta. \end{aligned} \tag{22}$$

On the other hand, for all $x\in D(A)\subset X$, we notice that

$$\begin{aligned} &\lambda^{\alpha-1}(\lambda^\alpha+A)^{-1}x \\ &= \int_0^\infty \lambda^{\alpha-1}e^{-\lambda^\alpha s}T(s)\,x\,ds \\ &= \int_0^\infty \alpha(\lambda t)^{\alpha-1}e^{-(\lambda t)^\alpha}T(t^\alpha)\,x\,dt \\ &= \int_0^\infty -\frac{1}{\lambda}\frac{d}{dt}\left[e^{-(\lambda t)^\alpha}\right]T(t^\alpha)\,dt \\ &= \int_0^\infty\int_0^\infty\theta\psi_\alpha(\theta)e^{-\lambda t\theta}T(t^\alpha)\,x\,d\theta\,dt \\ &= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty\psi_\alpha(\theta)T\left(\frac{t^\alpha}{\theta^\alpha}\right)x\,d\theta\right]dt \\ &= \int_0^\infty e^{-\lambda t}\left[\int_0^\infty\phi_\alpha(\theta)T(t^\alpha\theta)\,x\,d\theta\right]dt. \end{aligned} \tag{23}$$

Thus,

$$\mathcal{S}_\alpha(t) = \int_0^\infty\phi_\alpha(\theta)T(t^\alpha\theta)\,d\theta. \tag{24}$$

□

Remark 10. If $M_\alpha = \sup_{0\le t\le T}\|T(t)\|$, then from Lemma (9), we have

$$\|\mathcal{S}_{\alpha,k,l}(t)\|\le M_\alpha^{k+1-l}, \qquad \overline{S_{\alpha,k,l}}(t)\le M_\alpha^{k-l}. \tag{25}$$

4. Main Results

In this section, we will present the main results of this paper. The mild solution will be understood in the sense of Definition 7. To investigate the uniqueness of the mild solution, we require the following assumptions.

(H_1) The functions $F, G : I \times \mathscr{B} \to X$ and $I_i : \mathscr{B} \to X$, $i = 1, \ldots, m$, are continuous and satisfy the following conditions

(i) For every $x : (-\infty, T] \to X$ such that $x_0 = \varphi$ and $x|_I \in \mathscr{PC}$, the function $t \to G(t, x_t)$ is strongly measurable and the function $t \to F(t, x_t)$ belongs to $\mathscr{PC}$.

(ii) There exist a positive constant $\beta \in (0,1)$, and a function $\mu_1 \in L^{+\infty}(I, R^+)$ as well as $L_F, L, i = 1, \ldots, m$, such that F is X_β-valued, $A^\beta F : I \times \mathscr{B} \to X$ is continuous and

$$\begin{aligned} &\left\|G(t,\psi_1) - G(t,\psi_2)\right\| \\ &\quad \le \mu_1(t)\|\psi_1 - \psi_2\|_{\mathscr{B}}, \quad \psi_1, \psi_2 \in \mathscr{B},\ t \in I, \\ &\left\|A^\beta F(t,\psi_1) - A^\beta F(t,\psi_2)\right\| \\ &\quad \le L_F\|\psi_1 - \psi_2\|_{\mathscr{B}}, \quad \psi_1, \psi_2 \in \mathscr{B},\ t \in I, \\ &\left\|I_i(\psi_1) - I_i(\psi_2)\right\| \le L\|\psi_1 - \psi_2\|_{\mathscr{B}}, \quad \psi_1, \psi_2 \in \mathscr{B}. \end{aligned} \tag{26}$$

(H_2) The function $\zeta(t) : I \to R^+$ is defined by

(i)

$$\begin{aligned} \zeta(t) &= K_\sigma\Bigg[L_F\left\|A^{(-\beta)}\right\|\left(1 + \frac{2M_\alpha\left(M_\alpha^{m+1} - 1\right)}{M_\alpha - 1}\right) \\ &\quad + \frac{M_\alpha\left(M_\alpha^{m+1} - 1\right)L}{M_\alpha - 1} + \left(1 + \frac{M_\alpha\left(M_\alpha^{m+1} - 1\right)}{M_\alpha - 1}\right) \\ &\quad \times \left(\frac{C_{1-\beta}L_F\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}t^{\alpha\beta}\right) + \frac{M_\alpha\|\mu_1\|_{L^\infty(I,R^+)}}{\Gamma(\alpha+1)}t^\alpha\Bigg], \end{aligned} \tag{27}$$

when $M_\alpha \neq 1$, or

(ii)

$$\begin{aligned} \zeta(t) &= K_\sigma\Bigg[L_F\left\|A^{(-\beta)}\right\|(1 + 2m) + mL + (1+m) \\ &\quad \times \left(\frac{C_{1-\beta}L_F\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}t^{\alpha\beta} + \frac{\|\mu_1\|_{L^\infty(I,R^+)}}{\Gamma(\alpha+1)}t^\alpha\right)\Bigg], \end{aligned} \tag{28}$$

when $M_\alpha = 1$. Here, $K_\sigma = \sup_{s\in I} K(s)$ and satisfies $0 < \zeta(t) \le \gamma < 1$, for all $t \in I$.

Remark 11. Let $x : (-\infty, T] \to X$ be such that $x_0 = \varphi$ and $x|_I \in \mathscr{PC}$, and assume that H_1 holds. We have the following estimations:

$$\begin{aligned} &\left\|AT_\alpha(t-s)F(s,x_s)\right\| \\ &\quad = \left\|A^{1-\beta}T_\alpha(t-s)A^\beta F(s,x_s)\right\| \\ &\quad \le \Bigg\|\left[\alpha\int_0^\infty \theta\phi_\alpha(\theta)(t-s)^{\alpha-1}A^{(1-\beta)} \right. \\ &\qquad \left. \times T\left((t-s)^\alpha\theta\right)d\theta\right]A^\beta F(s,x_s)\Bigg\| \\ &\quad \le \alpha C_{1-\beta}(t-s)^{\alpha\beta-1} \\ &\qquad \times \int_0^\infty \theta^\beta\phi_\alpha(\theta)\,d\theta\left\|A^\beta F(s,x_s)\right\|. \end{aligned} \tag{29}$$

On the other hand, from $\int_0^\infty \theta^{-q}\psi_\alpha(\theta)d\theta = \Gamma(1+q/\alpha)/\Gamma(1+q)$, for all $q \in [0,1]$ (see [44]), we have

$$\begin{aligned} &\int_0^\infty \theta^\beta\phi_\alpha(\theta)\,d\theta \\ &\quad = \int_0^\infty \frac{1}{\theta^{\beta\alpha}}\psi_\alpha(\theta)\,d\theta \\ &\quad = \frac{\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)}. \end{aligned} \tag{30}$$

Then, by (29) and (30), it is easy to see that

$$\begin{aligned} &\left\|AT_\alpha(t-s)F(s,x_s)\right\| \\ &\quad \le \frac{\alpha C_{1-\beta}\Gamma(1+\beta)}{\Gamma(1+\alpha\beta)(t-s)^{1-\alpha\beta}}\left\|A^\beta F(s,x_s)\right\|. \end{aligned} \tag{31}$$

It is obvious that the function $\theta \to AT_\alpha(t-\theta)F(\theta, x_\theta)$ is integrable on $[0,t)$ for every $t > 0$. Similarly, it can be verified that the function $\theta \to AT_\alpha(t-\theta)G(\theta, x_\theta)$ is integrable on $[0,t)$ for every $t > 0$.

Theorem 12. *Under the assumptions* (H_1) *and* (H_2)*, the system* (1) *has a unique mild solution.*

Proof. On the metric space $\mathscr{BPC} = \{u : (-\infty, T] \to X, u_0 = \varphi, u|_I \in \mathscr{PC}\}$ endowed with the metric $d(u,v) = \|u - v\|_{\mathscr{PC}}$, we define the operator $\Gamma : \mathscr{BPC} \to \mathscr{BPC}$ by

$$\Gamma u(t) = \begin{cases} \varphi(t), & t \le 0 \\ \mathcal{S}_{\alpha,k,0}(t)(\varphi(0) + F(0,\varphi)) - F(t,u_t) \\ \quad + \sum_{0<t_{l+1}<t, 0\le l\le k} \overline{\mathcal{S}_{\alpha,k,l}}(t) \\ \quad \times \int_{t_l}^{t_{l+1}} AT_\alpha(t_{l+1}-s) F(s,u_s)\, ds \\ \quad + \int_{t_k}^{t} AT_\alpha(t-s) F(s,u_s)\, ds \\ \quad + \sum_{0<t_{l+1}<t, 0\le l\le k} \overline{\mathcal{S}_{\alpha,k,l}}(t) \\ \quad \times \int_{t_l}^{t_{l+1}} T_\alpha(t_{l+1}-s) G(s,u_s)\, ds \\ \quad + \int_{t_k}^{t} T_\alpha(t-s) G(s,u_s)\, ds \\ \quad + \sum_{0<t_{l+1}<t, 0\le l\le k} \overline{\mathcal{S}_{\alpha,k,l}}(t) \\ \quad \times [I_{l+1}(u_{t_{l+1}}) + \Delta F(t_{l+1}, u_{t_{l+1}})] & t \in I. \end{cases} \tag{32}$$

From Remark 11, we know that $s \to T_\alpha(t-s)G(s,u_s)$ and $s \to T_\alpha(t-s)AF(s,u_s)$ are integrable on $[0,t)$ for every $t \in I$. Thus, Γ is well defined with values in $\mathscr{PC}$. Let $u, v \in \mathscr{BPC}$. Using $\|\Delta F(t_i,u_{t_i}) - \Delta F(t_i, v_{t_i})\| \le 2L_F K_\sigma \|A^{(-\beta)}\| \|(u-v)|_I\|_{\mathscr{PC}}$, we get

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\| \\
&\quad \le \|F(t,u_t) - F(t,v_t)\| \\
&\qquad + \sum_{0<t_{l+1}<t, 0\le l\le k} \left\|\overline{\mathcal{S}_{\alpha,k,l}}(t)\right\| \\
&\qquad \times \int_{t_l}^{t_{l+1}} \|AT_\alpha(t_{l+1}-s) F(s,u_s) \\
&\qquad\qquad - AT_\alpha(t_{l+1}-s) F(s,v_s)\|\, ds \\
&\qquad + \int_{t_k}^{t} \|AT_\alpha(t-s)(F(s,u_s) - F(s,v_s))\|\, ds \\
&\qquad + \sum_{0<t_{l+1}<t, 0\le l\le k} \left\|\overline{\mathcal{S}_{\alpha,k,l}}(t)\right\| \\
&\qquad \times \int_{t_l}^{t_{l+1}} \|T_\alpha(t_{l+1}-s)(G(s,u_s) - G(s,v_s))\|\, ds \\
&\qquad + \int_{t_k}^{t} (t-s)^{(\alpha-1)} \\
&\qquad\quad \times \|T_\alpha(t-s)(G(s,u_s) - G(s,v_s))\|\, ds \\
&\qquad + \sum_{0<t_{l+1}<t, 0\le l\le k} \left\|\overline{\mathcal{S}_{\alpha,k,l}}(t)\right\| \\
&\qquad \times [\|I_{l+1}(u_{t_{l+1}}) - I_{l+1}(v_{t_{l+1}})\| \\
&\qquad\quad + \|\Delta F(t_{l+1}, u_{t_{l+1}}) - \Delta F(t_{l+1}, v_{t_{l+1}})\|] \\
&\quad \le \left\|A^{(-\beta)}\right\| L_F \|u_t - v_t\|_{\mathscr{B}} \\
&\qquad + \sum_{0<t_{l+1}<t, 0\le l\le k} \left\|\overline{\mathcal{S}_{\alpha,k,l}}(t)\right\| \\
&\qquad\quad \times \int_{t_l}^{t_{l+1}} (t_{l+1}-s)^{(\alpha\beta-1)} \\
&\qquad\qquad \times \frac{\alpha C_{1-\beta} L_F \Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} \|u_s - v_s\|_{\mathscr{B}} ds \\
&\qquad + \int_{t_k}^{t} (t-s)^{\alpha\beta-1} \frac{\alpha C_{1-\beta} L_F \Gamma(1+\beta)}{\Gamma(1+\alpha\beta)} \|u_s - v_s\|_{\mathscr{B}} ds \\
&\qquad + \alpha M_\alpha \sum_{0<t_{l+1}<t, 0\le l\le k} \left\|\overline{\mathcal{S}_{\alpha,k,l}}(t)\right\| \int_{t_l}^{t_{l+1}} (t_{l+1}-s)^{\alpha-1} \\
&\qquad \times \mu_1(s) \|u_s - v_s\|_{\mathscr{B}} \left(\int_0^\infty \theta \phi_\alpha(\theta)\, d\theta\right) ds \\
&\qquad + \alpha M_\alpha \int_{t_k}^{t} (t-s)^{\alpha-1} \mu_1(s) \\
&\qquad\qquad \times \|u_s - v_s\|_{\mathscr{B}} \left(\int_0^\infty \theta \phi_\alpha(\theta)\, d\theta\right) ds \\
&\qquad + \sum_{0<t_{l+1}<t, 0\le l\le k} \left\|\overline{\mathcal{S}_{\alpha,k,l}}(t)\right\| \\
&\qquad \times \left(L\|u_{t_i} - v_{t_i}\|_{\mathscr{B}} + 2L_F K_\sigma \left\|A^{(-\beta)}\right\| \|(u-v)|_I\|_{\mathscr{PC}}\right) \\
&\quad \le \left\|A^{(-\beta)}\right\| L_F \|u_t - v_t\|_{\mathscr{B}} \\
&\qquad + \left(1 + \frac{M_\alpha (M_\alpha^{m+1} - 1)}{M_\alpha - 1}\right) \frac{C_{1-\beta} L_F \Gamma(1+\beta) t^{\alpha\beta}}{\beta \Gamma(1+\alpha\beta)} \\
&\qquad \times \|u_s - v_s\|_{\mathscr{B}} \\
&\qquad + \frac{M_\alpha t^\alpha \|\mu_1\|_{L^\infty(I,R^+)}}{\Gamma(\alpha+1)} \left(1 + \frac{M_\alpha (M_\alpha^{m+1} - 1)}{M_\alpha - 1}\right) \\
&\qquad \times \|u_s - v_s\|_{\mathscr{B}} \\
&\qquad + \frac{M_\alpha (M_\alpha^{m+1} - 1)}{M_\alpha - 1} \\
&\qquad \times \left(L\|u_{t_i} - v_{t_i}\|_{\mathscr{B}} + 2L_F K_\sigma \left\|A^{(-\beta)}\right\| \|(u-v)|_I\|_{\mathscr{PC}}\right),
\end{aligned} \tag{33}$$

for all $t \in [0, T]$. □

Remark 13. The above inequality makes use of

$$\int_0^\infty \theta\phi_\alpha(\theta)\,d\theta = \frac{1}{\Gamma(1+\alpha)}, \quad (34)$$

which can be obtained directly from (30).

Remark 14. If $M_\alpha = 1$ in the above inequality, we replace $(M_\alpha(M_\alpha^{m+1}-1))/(M_\alpha-1)$ by m.

Therefore, it can be deduced that

(i) when $M_\alpha \neq 1$,

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\| \\
&\le K_\sigma\left[L_F\left\|A^{(-\beta)}\right\|\left(1+\frac{2M_\alpha(M_\alpha^{m+1}-1)L_F}{M_\alpha-1}\right)\right. \\
&\quad + \frac{M_\alpha(M_\alpha^{m+1}-1)L}{M_\alpha-1} + \left(1+\frac{M_\alpha(M_\alpha^{m+1}-1)}{M_\alpha-1}\right) \\
&\quad \left.\times\left(\frac{C_{1-\beta}L_F\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}t^{\alpha\beta} + \frac{M_\alpha\|\mu_1\|_{L^\infty(I,R^+)}}{\Gamma(\alpha+1)}t^\alpha\right)\right] \\
&\times\|(u-v)|_I\|_{\mathscr{PC}}, \quad \text{and}
\end{aligned} \quad (35)$$

(ii) when $M_\alpha = 1$,

$$\begin{aligned}
&\|\Gamma u(t) - \Gamma v(t)\| \\
&\le K_\sigma\left[L_F\left\|A^{(-\beta)}\right\|(1+2m) + mL + (1+m)\right. \\
&\quad \left.\times\left(\frac{C_{1-\beta}L_F\Gamma(1+\beta)}{\beta\Gamma(1+\alpha\beta)}t^{\alpha\beta} + \frac{\|\mu_1\|_{L^\infty(I,R^+)}}{\Gamma(\alpha+1)}t^\alpha\right)\right] \\
&\times\|(u-v)|_I\|_{\mathscr{PC}}.
\end{aligned} \quad (36)$$

Thus,

$$\|\Gamma u(t) - \Gamma v(t)\| \le \zeta(t)\,\|(u-v)|_I\|_{\mathscr{PC}}. \quad (37)$$

Therefore, in view of the contraction mapping principle, assumption (H_2) implies that Γ has a unique fixed point on $\mathscr{BPC}$. Thus, the system (1) has a unique mild solution.

5. An Example

In this section, we present an example to further illustrate the applications of the results given in Section 4.

Let $X = L^2([0,\pi])$ be the space of the square integrable functions and $\mathscr{B} = \mathscr{PC}_r \times L^2(g,X)$. We investigate the operator $-A : \mathscr{D}(-A) \subset X \to X$ defined by $(-A)x = x''$, where

$$\mathscr{D}(-A) = \left\{x \in X : x'' \in X,\ x(0) = x(\pi) = 0\right\}. \quad (38)$$

We note that $-A$ is the infinitesimal generator of an analytic compact semigroup $\{T(t)\}_{t\ge0}$ on X. In particular, we have that $(T(t)_{t\ge0})$ is a uniformly stable semigroup with $\|T(t)\| \le e^{-t}$ for all $t \ge 0$. Moreover,

$$D(A^\beta) = \left\{x : x \in X,\ \sum_{i=1}^n i^{2\beta}\langle x, z_i\rangle z_i \in X\right\}. \quad (39)$$

It follows from the above properties that analytic semigroup $(T(t))_{t\ge0}$ satisfies

$$\left\|(-A)^{1/2}T(t)\right\|_X \le \frac{e^{-t/2}t^{-1/2}}{\sqrt{2}}, \quad \left\|(-A)^{-1/2}\right\| = 1, \quad \text{for } t \ge 0. \quad (40)$$

Consider the impulsive fractional partial neutral differential equation system:

$$\begin{aligned}
&D_t^\alpha\left[\omega(t,\xi) + \int_{-\infty}^t\int_0^\pi b(s-t,\eta,\xi)\,\omega(\xi,s)\,d\eta\,ds\right] \\
&\quad - \frac{\partial^2}{\partial\xi^2}\omega(t,\xi) = \mathscr{N}(t,\omega(t,\xi)), \\
&\quad \omega(t,0) = \omega(t,\pi) = 0, \quad t \in I = [0,T], \\
&\quad \omega(\tau,\xi) = \varphi(\tau,\xi), \quad \tau \le 0,\ \xi \in [0,\pi], \\
&\Delta\omega(t_i^+,\cdot) - \omega(t_i^-,\cdot) = \int_0^\pi p_i(\xi,\omega(t_i,s))\,ds, \\
&\qquad i = 1,\dots,m,
\end{aligned} \quad (41)$$

where (t_i) is a strictly increasing sequence of positive real numbers. To treat this system, we assume that the following conditions hold.

$(a)^*$ The function $\mathscr{N} : R\times R \to R$ is continuous and there is a continuous integrable function $\mu_1 : R \to [0,\infty)$ such that

$$|\mathscr{N}(t,x)| \le \mu_1(t)\,|x|, \quad t \in [0,T],\ x \in R. \quad (42)$$

$(b)^*$ The function $b(s,\eta,\xi), (\partial/\partial\xi)b(s,\eta,\xi)$ is integrable, $b(s,\eta,\pi) = b(s,\eta,0)$, and

$$\begin{aligned}
&L_F \\
&:= \sup\left\{\int_0^\pi\int_{-\infty}^0\int_0^\pi\left(\frac{1}{g(s)}\frac{\partial^i b(s,\eta,\xi)}{\partial\xi^i}d\eta\,ds\,d\xi\right)^2 : i = 0,1\right\} \\
&< +\infty.
\end{aligned} \quad (43)$$

$(c)^*$ The function $p_i : [0,\pi]\times R \to R$, $(i = 1,\dots,m)$ is continuous and there exists a positive constant L such that

$$\begin{aligned}
&\left[p_i(\xi,s) - p_i(\xi,\bar{s})\right] \le L|s-\bar{s}|, \\
&\qquad \xi \in [0,\pi],\ s,\bar{s} \in R.
\end{aligned} \quad (44)$$

We now define the functions $F, G : [0,\alpha] \times \mathscr{B} \rightarrow X$ and $I_i : X \rightarrow X$ by

$$\begin{aligned} F(t,\phi)(\xi) &= \int_{-\infty}^{0} \int_{0}^{\pi} b(s,\eta,\xi)\,\phi(s,\eta)\,d\eta\,ds, \\ &\qquad t \in [0,T],\ \xi \in [0,\pi], \\ G(t,\phi)(\xi) &= \mathscr{N}(t,\phi(0,s)), \quad t \in [0,T],\ \xi \in [0,\pi], \\ I_i(\phi)(\xi) &= \int_0^{\pi} p_i(\xi,\phi(0,s))\,ds, \quad i \in N,\ \xi \in [0,\pi], \end{aligned} \tag{45}$$

respectively. It is easy to see that problem (41) can be modeled as the abstract impulsive Cauchy problem (1).

Theorem 15. *Assume that the conditions $(a)^*$, $(b)^*$, and $(c)^*$ hold and*

$$\left[\int_0^{\pi} \left|p_i(\xi,0)\right|^2 d\xi\right]^{1/2} < \infty. \tag{46}$$

If

$$\begin{aligned} K_\sigma \Bigg[L_F(1+2m) &+ \frac{(m+1)\,C_{1-\beta}L_F}{\beta\Gamma(1+\alpha\beta)} t^{\alpha\beta} \\ &+ \frac{(m+1)\,\|\mu_1\|_{L^\infty(I,R^+)}}{\Gamma(\alpha+1)} t^{\alpha} + \pi\sqrt{\pi}\, mL \Bigg] < 1, \end{aligned} \tag{47}$$

for all $t \in I$, then there is a mild solution of the impulsive system (41).

Proof. It follows from $(b)^*$ that F is $X_{1/2}$-valued and $F : I \times \mathscr{B} \rightarrow X_{1/2}$ is continuous. Moreover, $A^{1/2}F(t,\cdot)$ is a bounded linear operator, and as such $\|A^{1/2}F(t,\cdot)\| \le L_F$ for all $t \in I$. By $(a)^*$, one can easily conclude that the function $G(t,\cdot) : \mathscr{B} \rightarrow X$ is continuous for all $t \in [0,T]$, and $G(\cdot,\phi) : [0,\infty) \rightarrow X$ is a strongly measurable function for each $\phi \in \mathscr{B}$. In addition, for all $t \in [0,T]$ and $\phi \in \mathscr{B}$, we have

$$\begin{aligned} \|G(t,\phi)\|_X^2 &= \int_0^{\pi} |\mathscr{N}(t,\phi(0,\xi))|^2 d\xi \\ &\le \int_0^{\pi} \mu_1^2(t)\,|\phi(0,\xi)|^2 d\xi \\ &\le \mu_1^2(t)\,\|\phi\|_{\mathscr{B}}^2. \end{aligned} \tag{48}$$

It follows from condition $(c)^*$ that

$$\begin{aligned} &|I_i(\phi)(\xi) - I_i(\psi)(\xi)| \\ &\quad \le \int_0^{\pi} \left|p_i(\xi,\phi(0,s)) - p_i(\xi,\psi(0,s))\right| ds \\ &\quad \le L \int_0^{\pi} |\phi(0,s) - \psi(0,s)|\,ds \\ &\quad \le \pi L \|\phi - \psi\|_{\mathscr{B}}, \end{aligned} \tag{49}$$

for all $\varphi, \psi \in \mathscr{B}$. This implies that $\|I_i(\phi) - I_i(\psi)\|_X \le \pi\sqrt{\pi}L\|\phi - \psi\|_{\mathscr{B}}$. The proof is complete since the existence of a mild solution is a consequence of Theorem 15. □

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